Sample Question Paper - 11
Mathematics (041)
Class- XII, Session: 2021-22
TERM II

Time Allowed: 2 hours **Maximum Marks: 40**

General Instructions:

1. This question paper contains three sections – A, B and C. Each part is compulsory.
2. Section - A has 6 short answer type (SA1) questions of 2 marks each.
3. Section – B has 4 short answer type (SA2) questions of 3 marks each.
4. Section - C has 4 long answer-type questions (LA) of 4 marks each.
5. There is an internal choice in some of the questions.
6. Q 14 is a case-based problem having 2 sub-parts of 2 marks each.

Section A

1. Evaluate: $\int \frac{x+1}{x(x+\log x)} dx$ **[2]**

OR

Evaluate the integral: $\int \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$

2. Find the general solution of the differential equation: $\left(1+x^2\right) \frac{dy}{dx}+2xy=\frac{1}{(1+x^2)}$ **[2]**

3. For any vector $\vec{a}$, prove that $|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=2|\vec{a}|^2$ **[2]**

4. Find the direction cosines of the line $\frac{x-2}{2}=\frac{2y-5}{-3}$, z = -1. Also, find the vector equation of the line. **[2]**

5. The probability that a person will get an electric contract is $\frac{2}{5}$ and the probability that he will not get plumbing contract is $\frac{4}{7}$.If the probability of getting at least one contract is $\frac{2}{3}$, what is the probability that he will get both? **[2]**

6. There are three urns A, B, and C. Urn A contains 4 red balls and 3 black balls. Urn B contains 5 red balls and 4 black balls. Urn C contains 4 red and 4 black balls. One ball is drawn from each of these urns. What is the probability that 3 balls drawn consist of 2 red balls and a black ball? **[2]**

Section B

7. Evaluate $\int e^{-3x}\cos^3 x dx$ **[3]**

8. Solve the differential equation: $(x^2 + 3xy + y^2)\, dx - x^2\, dy = 0$ **[3]**

OR

Verify that $y = Ae^{ax} \cos bx + Be^{ax} \sin bx$, where A and B are arbitrary constants, is the general solution of the differential equation $\frac{d^2y}{dx^2}-2a\frac{dy}{dx} + (a^2 + b^2)y = 0$.

9. If $\vec{a}=\hat{i}+2\hat{j}+\hat{k}, \vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{i}-4\hat{j}-5\hat{k}$, then find a unit vector perpendicular to both of the vectors $(\vec{a}-\vec{b})$ and $(\vec{c}-\vec{b})$. **[3]**

10. Find the intercepts made on the coordinate axes by the plane 2x + y - 2z = 3 and find also the direction cosines of the normal to the plane. **[3]**

OR

Show that the line whose vector equation is $\vec{r}=2\hat{i}+5\hat{j}+7\hat{k}+\lambda(\hat{i}+3\hat{j}+4\hat{k})$ is parallel to the plane whose vector equation is $\vec{r}\cdot(\hat{i}+\hat{j}-\hat{k})=7$. Also, find the distance between them.

Section C

11. Evaluate: $\int\frac{x^2}{(x^2+6x-3)}dx$ **[4]**

12. Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$. **[4]**

OR

Using integration find the area of the region bounded by the curve $y=\sqrt{4-x^2}$, $x^2 + y^2 - 4x = 0$ and the x-axis.

13. Show that the lines $\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$ are coplanar. Also find the equation of the plane containing them. **[4]**

CASE-BASED/DATA-BASED

14. Kamal is a good card player. He plays many magic tricks on cards as well. He has a deck of 52 cards. He shuffled the cards well and drew five cards one by one, with replacement. **[4]**

Find the probability that

i. all five cards are diamonds.
ii. none is a diamond.

Solution

MATHEMATICS 041

Class 12 - Mathematics

Section A

1. Let I = $\int \frac{x+1}{x(x+\log x)} dx$ ...(i)

Now let (x + log x) = t then, we have

d(x + log x) = dt

$\Rightarrow \quad \left(1+\frac{1}{x}\right) dx = dt$

$\Rightarrow \quad \left(\frac{x+1}{x}\right) dx = dt$

$\Rightarrow \quad dx = \frac{x}{x+1} dt$

Putting (x + log x) = t and $dx = \frac{x}{x+1}$ in equation (i), we get

$I = \int \frac{x+1}{x \times t} \times \frac{x}{x+1} dt$

$= \int \frac{dt}{t}$

= log |t| + c

= log |x + logx| + c

$\Rightarrow$ I = log |x + log x| + c

OR

Let, $I = \int \left(\frac{\sin x+\cos x}{\sqrt{\sin 2x}}\right) dx$

Put, sin x - cos x = t

$\Rightarrow$ (cos x + sin x)dx = dt

Also $(\sin x - \cos x)^2 = t^2$

$\Rightarrow \sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$

$\Rightarrow 1 - t^2 = \sin(2x)$

$\therefore I = \int \frac{dt}{\sqrt{1-t^2}}$

$= \sin^{-1} t + C \quad \ldots\left[\int \frac{dt}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C\right]$

= $\sin^{-1}$(sin x - cos x) + C ($\because$ t = sin x - cos x)

2. Given $\left(1+x^2\right) \frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{1}{\left(1+x^2\right)^2}$

This is of the form $\frac{dy}{dx} + Py = Q$

where $P = \frac{2x}{1+x^2}$ and $Q = \frac{1}{\left(1+x^2\right)^2}$

This the given differential equation is linear

Now, $IF = e^{\int Pdx} \Rightarrow I \cdot F = e^{\int \frac{2xdx}{1+x^2}}$

$= e^{\log\left(1+x^2\right)} = 1 + x^2$

Therefore the solution is given by $y \cdot (1.F) = \int (1.F)Q + C$

$\Rightarrow y\left(1+x^2\right) = \int \left(1+x^2\right) \frac{1}{\left(1+x^2\right)^2} dx + C$

$\Rightarrow y\left(1+x^2\right) = \int \frac{dx}{1+x^2} + C$

$\Rightarrow$ y. $(1 + x^2) = \tan^{-1}$ x + C.

3. Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ ...(i)

Then,

$\vec{a} \times \hat{i} = \left(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}\right) \times \hat{i} = a_1(\hat{i} \times \hat{i}) + a_2(\hat{j} \times \hat{i}) + a_3(\hat{k} \times \hat{i}) = -a_2 \hat{k} + a_3 \hat{j}$

$\Rightarrow |\vec{a} \times \hat{i}|^2 = a_2^2 + a_3^2$

$\vec{a}\times\hat{j}=\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\times\hat{j}=a_1(\hat{i}\times\hat{j})+a_2(\hat{j}\times\hat{j})+a_3(\hat{k}\times\hat{j})=a_1\hat{k}-a_3\hat{i}$

$\Rightarrow|\vec{a}\times\hat{j}|^2=a_1^2+a_3^2$

and $\vec{a}\times\hat{k}=\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\times\hat{k}=a_1(\hat{i}\times\hat{k})+a_2(\hat{j}\times\hat{k})+a_3(\hat{k}\times\hat{k})=-a_1\hat{j}+a_2\hat{i}$

$\Rightarrow|\vec{a}\times\hat{k}|^2=a_1^2+a_2^2$

$\therefore|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=a_2^2+a_3^2+a_1^2+a_3^2+a_1^2+a_2^2$

$\Rightarrow|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=2\left(a_1^2+a_2^2+a_3^2\right)=2|\vec{a}|^2$ ....using (i)

4. The Cartesian equations of the given line are

$\frac{x-2}{2}=\frac{2y-5}{-3}$, z = -1

These above equations can be re-written as

$\frac{x-2}{2}=\frac{2y-5}{-3}=\frac{z+1}{0}$ or, $\frac{x-2}{2}=\frac{y-5/2}{-3/2}=\frac{z+1}{0}$

This shows that the given line passes through the point (2, $\frac{5}{2}$, - 1) and has direction ratios proportional to 2, -$\frac{3}{2}$,0. So, its direction cosines are

$\frac{2}{\sqrt{2^2+\left(-\frac{3}{2}\right)^2+0^2}},\frac{-3/2}{\sqrt{2^2+\left(-\frac{3}{2}\right)^2+0^2}},\frac{0}{\sqrt{2^2+\left(-\frac{3}{2}\right)^2+0^2}}$ or, $\frac{2}{5/2},\frac{-3/2}{5/2},0$

or, $\frac{4}{5},-\frac{3}{5},0$

The given line passes through the point having a position vector $\vec{a}=2\hat{i}+\frac{5}{2}\hat{j}-\hat{k}$ and is parallel to the vector $\vec{b}=2\hat{i}-\frac{3}{2}\hat{j}+0\hat{k}$.

Therefore, it's vector equation is

$\vec{r}=\left(2\hat{i}+\frac{5}{2}\hat{j}-\hat{k}\right)+\lambda\left(2\hat{i}-\frac{3}{2}\hat{j}+0\hat{k}\right)$

5. Consider the following events:

A = Person gets an electric contract, B = Person gets plumbing contract .

Therefore,we have,

P(A) =$\frac{2}{5}$, P ($\overline{B}$) = $\frac{4}{7}$ and P (A ∪ B) = $\frac{2}{3}$

By addition theorm of probability, we have,

P (A ∪ B) = P (A) + P (B) - P (A ∩ B)

$\Rightarrow\frac{2}{3}=\frac{2}{5}+\left(1-\frac{4}{7}\right)$ - P (A ∩ B)

$\Rightarrow$ P (A ∩ B) = $\frac{2}{5}+\frac{3}{7}-\frac{2}{3}=\frac{17}{105}$

6. We are given that,

Urn A (4R + 3B)

Urn B (5R+ 4B)

Urn C (4R + 4B)

Required probability is given by,

P(two red and one black) = P(Red from urn A) × (Black from urn B) × P(Red from urn C) + P(Red from urn A) × (Red from urn B) × P(Black from urn C) + P(Black from urn A) × (Red from urn B) × P(Red from urn C)

$=\frac{3}{7}\times\frac{5}{9}\times\frac{4}{8}+\frac{4}{7}\times\frac{4}{9}\times\frac{4}{8}+\frac{4}{7}\times\frac{5}{9}\times\frac{4}{8}$

$=\frac{5}{42}\times\frac{16}{126}\times\frac{20}{126}$

$=\frac{15+16+20}{126}$

$=\frac{51}{126}=\frac{17}{42}$

Section B

7. Given integral is: $\int e^{-3x}\cos^3 x dx$

Using trigonometric identity cos 3x = 4 cos^3 x - 3 cos x

$\Rightarrow\int e^{-3x}\cos^3 x dx=\frac{1}{4}\int e^{-3x}(\cos 3x+3\cos x)dx$

$\Rightarrow\frac{1}{4}\int e^{-3x}(\cos 3x+3\cos x)dx=\frac{1}{4}\int e^{-3x}\cos 3x dx+\frac{3}{4}\int e^{-3x}\cos x dx$ ...(i)

Using a general formula i.e.

$\Rightarrow\int e^{ax}\cos bx dx=\frac{e^{ax}}{a^2+b^2}$ (a cos bx + b sin bx)

$\Rightarrow \int e^{-3x} \cos 3x dx = \frac{e^{-3x}}{(-3)^2+3^2}$ ((-3) cos 3x + 3 sin 3x)

$\Rightarrow \frac{e^{-3x}}{(-3)^2+3^2}$ ((-3) cos 3x + 3 sin 3x) = $\frac{e^{-3x}}{6}$ (sin 3x - cos 3x) .....(ii)

$\Rightarrow \int e^{-3x} \cos x dx = \frac{e^{-3x}}{(-3)^2+1^2}$ ((-3) cos 3x + 3 sin 3x) = $\frac{e^{-3x}}{10}$ (sin x - 3 cos x)

$= \frac{e^{-3x}}{10}$ (sin x - 3 cos x) .....(iii)

On putting (ii) and (iii) in (i)

$\Rightarrow \frac{1}{4}\int e^{-3x} \cos 3x dx + \frac{3}{4}\int e^{-3x} \cos x dx = \frac{e^{-3x}}{4\times 6}(\sin 3x - \cos 3x) + \frac{3e^{-3x}}{4\times 10}$ (sin x - cos x)

$\Rightarrow \int e^{-3x} \cos^3 x dx = e^{-3x}\left\{\frac{(\sin 3x - \cos 3x)}{24} + \frac{3(\sin x - 3\cos x)}{40}\right\} + C$

8. The given differential equation is,

$(x^2 + 3xy + y^2)\, dx - x^2\, dy = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x^2+3xy+y^2}{x^2}$

This is a homogeneous differential equation

Putting y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$v + x\frac{dv}{dx} = \frac{x^2+3vx^2+v^2x^2}{x^2}$

$\Rightarrow x\frac{dv}{dx} = 1 + 3v + v^2 - v$

$\Rightarrow x\frac{dv}{dx} = 1 + v^2 + 2v$

$\Rightarrow \frac{1}{1+v^2+2v}dv = \frac{1}{x}dx$

Integrating both sides, we get

$\int \frac{1}{1+v^2+2v}dv = \int \frac{1}{x}dx$

$\Rightarrow \int \frac{1}{(1+v)^2}dv = \int \frac{1}{x}dx$

$\Rightarrow -\frac{1}{(1+v)} = \log|x| + C$

$\Rightarrow \log|x| + \frac{1}{(1+v)} = -C$

Putting $v = \frac{y}{x}$, we get

$\therefore \log|x| + \frac{x}{(x+y)} = C_1$

where, $C_1 = -C$

Hence, $\log|x| + \frac{x}{(x+y)} = C_1$ is the required solution.

OR

We have, $y = Ae^{ax}\cos bx + Be^{ax}\sin bx$ ...(i)

Differentiating (i) on both sides w.r.t. x, we have,

$\frac{dy}{dx}$ = A · [e^{ax} {-b sin bx) + ae^{ax} cos bx} + B · (e^{ax} (b cos bx) + ae^{ax} sin bx}

$\Rightarrow \frac{dy}{dx}$ = a (Ae^{ax} cos bx + Be^{ax} sin bx} + b {-Ae^{ax} sin bx + Be^{ax} cos bx}

$\Rightarrow \frac{dy}{dx}$ = ay + be^{ax} {B cos bx - A sin bx} ...(ii) [using (i)]

Differentiating (ii) on both sides w.r.t. x, we have,

$\frac{d^2y}{dx^2} = a\frac{dy}{dx}$ + b [e^{ax} (-b B sin bx - b A cos bx) + (B cos bx - A sin bx) ae^{ax}]

$\Rightarrow \frac{d^2y}{dx^2} = a\frac{dy}{dx}$ - b^2 {Ae^{ax} cos bx + Be^{ax} sin bx} + a {be^{ax} (B cos bx - A sin bx)}

$\Rightarrow \frac{d^2y}{dx^2} = a\frac{dy}{dx} - b^2y + a\left(\frac{dy}{dx} - ay\right)$ [using (i) and (ii)]

$\Rightarrow \frac{d^2y}{dx^2} - 2a\frac{dy}{dx}$ + $(a^2 + b^2)$ y = 0, which is the required differential equation.

Hence,the given functional relation is a solution of the given differential equation.

9. According to the question,

$\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$,

$\vec{b} = 2\hat{i} + \hat{j}$ and

$\vec{c} = 3\hat{i} - 4\hat{j} - 5\hat{k}$

Now, $\vec{a} - \vec{b} = (i + 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j}) = -\hat{i} + \hat{j} + \hat{k}$

Now, $\vec{c} - \vec{b} = (3\hat{i} - 4\hat{j} - 5\hat{k}) - (2\hat{i} + \hat{j}) = \hat{i} - 5\hat{j} - 5\hat{k}$

Now, a vector perpendicular to $(\vec{a} - \vec{b})$ and $(\vec{c} - \vec{b})$ is given by

$$\overrightarrow{(a - \vec{b})} \times (\vec{c} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -5 & -5 \end{vmatrix}$$

$= \hat{i}(-5 + 5) - \hat{j}(5 - 1) + \hat{k}(5 - 1)$

$= \hat{i}(0) - \hat{j}(4) + \hat{k}(4)$

$= -4\hat{j} + 4\hat{k}$

Unit vector along $(\vec{a} - \vec{b}) \times (\vec{c} - \vec{b})$ is given by

$\frac{-4\hat{j}+4\hat{k}}{|-4\hat{j}+4\hat{k}|}$

$= \frac{-4\hat{j}+4\hat{k}}{\sqrt{(-4)^2+4^2}}$

$= \frac{-4\hat{j}+4\hat{k}}{\sqrt{32}}$

$= \frac{-4\hat{j}+4\hat{k}}{4\sqrt{2}}$

$= -\frac{\hat{j}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}$

10. According to question the given equation of the plane is 2x + y - 2z = 3

Dividing both sides by 3, we obtain

$\frac{2x}{3} + \frac{y}{3} + \frac{-2z}{3} = \frac{3}{3}$

$\Rightarrow \frac{x}{\left(\frac{3}{2}\right)} + \frac{y}{3} + \frac{z}{\left(\frac{-3}{2}\right)} = 1$......(1)

We know that the equation of the plane whose intercepts on the coordinate axes are

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ....(2)

Comparing (1) and (2), we get

$a = \frac{3}{2}; b = 3; c = \frac{-3}{2}$

Finding the direction cosines of the normal

The given equation of the plane is

2x + y - 2z = 3

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + \hat{j} - 2\hat{k}) = 3$

$\Rightarrow \vec{r} \cdot (2\hat{i} + \hat{j} - 2\hat{k}) = 3$ which is the vector equation of the plane.

Also, $|\vec{n}| = \sqrt{4 + 1 + 4} = 3$

So, the unit vector perpendicular to $\vec{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\quad \hat{i}+\hat{j}-2\hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$

So, the direction cosines of the normal to the plane are, $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$

OR

We know that line $\overrightarrow{\mathrm{r}} = \overrightarrow{\mathrm{a}} + \mathrm{k}\overrightarrow{\mathrm{b}}$ and plane $\overrightarrow{\mathrm{r}} \cdot \overrightarrow{\mathrm{n}} = \mathrm{d}$ is parallel if

$\overrightarrow{\mathrm{b}} \cdot \overrightarrow{\mathrm{n}} = 0$

Given, the equation of the line

$\overrightarrow{\mathrm{r}} = (2\hat{\mathrm{i}} + 5\hat{\mathrm{j}} + 7\hat{\mathrm{k}}) + \mathrm{k}(\hat{\mathrm{i}} + 3\hat{\mathrm{j}} + 4\hat{\mathrm{k}})$ and the equation of the plane is;

$\overrightarrow{\mathrm{r}} \cdot (\hat{\imath} + \hat{\jmath} - \hat{\mathrm{k}}) = 7,$

So, $\overrightarrow{\mathrm{b}} = \hat{1} + 3\hat{\jmath} + 4\hat{\mathrm{k}}$ and $\overrightarrow{\mathrm{n}} = \hat{\mathrm{i}} + \hat{\mathrm{j}} - \hat{\mathrm{k}}$

Now, $\overrightarrow{\mathrm{b}} \cdot \overrightarrow{\mathrm{n}}$

$= (\hat{\imath} + 3\hat{\jmath} + 4\hat{k})(\hat{\imath} + \hat{\jmath} - \hat{k})$

= 1 + 3 – 4 = 0

So, the line and the plane are parallel

We know that the distance (D) of a plane $\overrightarrow{\mathrm{r}} \cdot \overrightarrow{\mathrm{n}} = \mathrm{d}$ from a point $\overrightarrow{\mathrm{a}}$ is given by

$\mathrm{D} = \frac{\overrightarrow{\mathrm{a}} \cdot \overrightarrow{\mathrm{n}} - \mathrm{d}}{\mathrm{n}}$

$\vec{a} = (2\hat{\imath} + 5\hat{\jmath} + 7\hat{k})$

$D = \frac{(2\hat{\imath}+5\hat{\jmath}+7\hat{k})\cdot(\hat{\imath}+\hat{\jmath}-\hat{k})-7}{\sqrt{1^2+1^2+(-1)^2}}$

$D = \frac{2+5-7-7}{\sqrt{1+1+1}}$

$D = -\frac{7}{\sqrt{3}}$

Since the distance is always positive,

So, $D = \frac{7}{\sqrt{3}}$

Section C

11. To find: $\int \frac{x^2}{(x^2+6x-3)} dx$

We will use following Formula ;

i. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

ii. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$

Given equation can be rewritten as following:

$\Rightarrow \int \frac{x^2+(6x-3)-(6x-3)}{(x^2+6x-3)} dx$

$\Rightarrow \int \frac{(x^2+6x-3)-(6x-3)}{(x^2+6x-3)} dx$

$\Rightarrow x - \int \frac{6x-3}{x^2+6x-3} dx$

Let $I = \int \frac{6x-3}{x^2+6x-3} dx$ ...(ii)

Using partial fractions,

$(6x-3) = A\left(\frac{d}{dx}(x^2+6x-3)\right) + B$

6x - 3 = A (2x + 6) + B

Equating the coefficients of x,

6 = 2A

A = 3

Also, -3 = 6A + B

$\Rightarrow$ B = -21

Substituting in (I),

$\Rightarrow \int \frac{3(2x+6)-21}{(x^2+6x-3)} dx$

$\Rightarrow 3 \times \log|x^2+6x-3| + C_1 - 21 \int \frac{1}{(x+3)^2-(\sqrt{12})^2} dx$

$\Rightarrow 3 \times \log|x^2+6x-3| + C_1 - 21 \times \frac{1}{2\sqrt{12}} \times \log\left|\frac{x+3-\sqrt{12}}{x+3+\sqrt{12}}\right| + C_2$

$I = 3\log|x^2+6x-3| - \frac{7\sqrt{3}}{4} \times \log\left|\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right| + C$

Therefore,

$\int \frac{x^2}{(x^2+6x-3)} dx = x - 3\log|x^2+6x-3| + \frac{7\sqrt{3}}{4} \times \log\left|\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right| + c$

12. We have

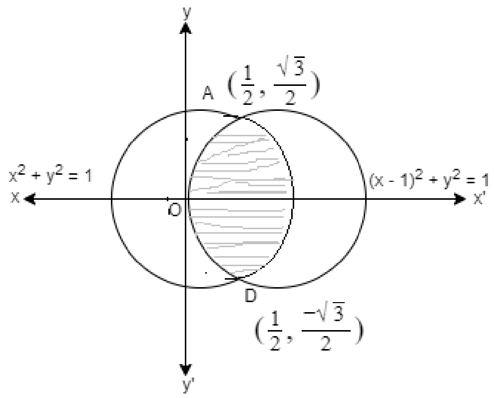
$x^2 + y^2 = 1$ ..... (i)

and $(x-1)^2 + y^2 = 1$ ...... (ii)

From (i) and (ii) we get point of Intersection as

$A\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), D\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$

As shown in fig.



Required Area , (Area OABD)

$= 2\left[\int_0^{1/2} y dx + \int_{1/2}^{1} y dx\right.$

$= 2\left[\int_0^{1/2} \sqrt{1-(x-1)^2}\, dx + \int_{1/2}^{1} \sqrt{1-x^2}\, dx\right.$

$= 2\left[\frac{(x-1)}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{1/2} + 2\left[\frac{x}{3}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right)_{1/2}^{1}$

$= \left[(x-1)\sqrt{1-(x-1)^2} + \sin^{-1}(x-1)\right]_0^{1/2} + \left[x\sqrt{1-x^2} + \sin^{-1}(x)\right]_{1/2}^{1}$

$= \left[-\frac{\sqrt{3}}{4} + 8\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right] + \left[\sin^{-1}(1) - \frac{\sqrt{3}}{4} - \sin^{-1}\left(\frac{1}{2}\right)\right]$

$= \left[-\frac{\sqrt{2}}{4} - \frac{\pi}{6} + \frac{\pi}{2}\right] + \left[\frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6}\right] = \left(\frac{2x}{3} - \frac{\sqrt{3}}{2}\right)$ sq. units

OR

The given curves are $y = \sqrt{4-x^2}$ and $x^2 + y^2 - 4x = 0$,

Now, $y = \sqrt{4-x^2} \Rightarrow x^2 + y^2 = 4$...(i)

This represents a circle with centre O(0,0) and radius =2 units.

Also,

$x^2 + y^2 - 4x = 0 \Rightarrow (x - 2)^2 + y^2 = 4$ ...(ii)

This represents a circle with centre B(2, 0) and radius = 2 units

Solving (i) and (ii), we get

$(x - 2)^2 = x^2$

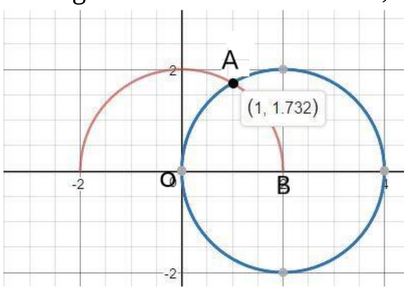
$\Rightarrow x^2 - 4x + 4 = x^2$

$\Rightarrow x = 1$

$\therefore y^2 = 3 \Rightarrow y = \pm\sqrt{3}$

Thus, the given circles intersect at $A(1, \sqrt{3})$ and $C(1, -\sqrt{3})$

A rough sketch of the curves is,



$\therefore$ Required area

= Area of shaded region OABO

$= \int_0^1 \sqrt{4-(x-2)^2}\, dx + \int_1^2 \sqrt{4-x^2}\, dx$

$= \left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$

$+ \left[\frac{1}{2}x\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$

$= \left[-\frac{\sqrt{3}}{2} + 2\sin^{-1}\left(-\frac{1}{2}\right)\right] - \left[0 + 2\sin^{-1}(-1)\right]$

$+ \left(0 - \frac{1}{2}\sqrt{3}\right) + 2\left[\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)\right]$

$= -\frac{\sqrt{3}}{2} - 2 \times \frac{\pi}{6} + 2 \times \frac{\pi}{2} - \frac{\sqrt{3}}{2} + 2 \times \frac{\pi}{2} - 2 \times \frac{\pi}{6}$
$= -\sqrt{3} + 2\pi - \frac{2\pi}{3}$
$\therefore A = \left(\frac{4\pi}{3} - \sqrt{3}\right)$ Square units

13. The given equations of lines are

$\frac{x-(a-d)}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-(a+d)}{\alpha+\delta}$ ...(i)

$\frac{x-(b-c)}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-(b+c)}{\beta+\gamma}$ ...(ii)

We know that the lines

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$

are coplanar , therefore$\Leftrightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

Here, $x_1 = (a - d)$, $y_1 = a$, $z_1 = (a + d)$;

$x_2 = (b - c)$, $y_2 = b$, $z_2 = (b + c)$;

$a_1 = (\alpha - \delta), b_1 = \alpha, c_1 = (\alpha + \delta)$ and
$a_2 = (\beta - \gamma), b_2 = \beta, c_2 = (\beta + \gamma)$

$$\therefore \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} (b-c)-(a-d) & b-a & (b+c)-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}$$

$$= \begin{vmatrix} 2(b-a) & b-a & b+c-a+d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} \quad \{C_1 \to C_1 + C_3\}$$

Therefore, the given lines are coplanar.

Equation of the plane containing the given lines is given by

$$\begin{vmatrix} x-(a-d) & y-a & z-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_3 - 2C_2$, we get

$$\begin{vmatrix} x-z-2y & y-a & z-(a+d) \\ 0 & \alpha & \alpha+\delta \\ 0 & \beta & \beta+\gamma \end{vmatrix} = 0$$

$\Rightarrow (x+z-2y) - [\alpha(\beta+\gamma) - \beta(\alpha+\delta)] = 0$
$\Rightarrow (x+z-2y)(\alpha y - \beta\delta) = 0$
$\Rightarrow x + z - 2y = 0$

Therefore, the required equation of the plane is $x + z - 2y = 0$.

CASE-BASED/DATA-BASED

14. Let X represents the number of diamond cards among the five cards drawn. Since, the drawing card is with replacement, so the trials are Bernoulli trials. In a well-shuffled deck of 52 cards, there are 13 diamond cards.

Now, p = P(success) = P(a diamond card is drawn)

$= \frac{13}{52} = \frac{1}{4}$

and q=p(failure) $= 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

Thus, X has a binomial distribution with

n = 5 $p = \frac{1}{4}$ and $q = \frac{3}{4}$

Therefore, $P(X = r) = {}^nC_r p^r q^{n-r}$,

where r = 0,1, 2 3, 4, 5

$P(X = r) = {}^5C_r \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{5-r}$

i. P(all the five cards are diamonds)= P(X = 5)

$= {}^5C_5 p^5 q^0 = 1p^5 = \left(\frac{1}{4}\right)^5 = \frac{1}{1024}$

ii. P(none is a diamond) = P(X = 0)

$= {}^5C_0 p^0 q^5 = (q)^5$

$= \left(\frac{3}{4}\right)^5 = \frac{243}{1024}$